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Quantum mechanics on Riemannian manifold in Schwinger's quantization approach III

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Abstract. Using the extended Schwinger quantization approach, quantum mechanics on a Riemannian manifold M with the given action of an intransitive group of isometries is developed. It was shown that quantum mechanics can be determined unequivocally only on submanifolds of M where G acts simply transitively (orbits of G action). The remaining part of the degrees of freedom can be described unequivocally after introducing some additional assumptions. Being logically unmotivated, these assumptions are similar to the canonical quantization postulates. Besides this ambiguity which is of a geometrical nature there is an undetermined gauge field of the order of $\hbar$ (or higher), vanishing in the classical limit $\hbar \to 0$.

1 Introduction

The purpose of the present paper is to continue a series of works we devote to a generalization of Schwinger's quantization approach for the case of a Riemannian manifold with a group structure (determined by its metric).

In this paper we turn to the most problematic case (as regards the physical meaning) of the formulation of quantum mechanics on a Riemannian manifold M with the intransitive group of isometries G. Partially our considerations are based on our previous results [2,3]. But, in contradiction to these, the main feature of the present paper is the fact that the dimension of the manifold M is higher than the dimension of the group of isometries G. This leads to a decomposition of the degrees of freedom describing a point particle on M into two sets, related to the group G and to the quotient space M/G. The latter manifold does not have a global group structure in the general case. Due to this the quantum mechanics turns out to be completely defined only on submanifolds of M which are isomorphic to G (the orbits of the action of G on M). The dynamical equations for the other degrees of freedom, connected with M/G, can be unequivocally determined after introducing some new external assumptions expanding Schwinger's scheme, which are similar to canonical quantization postulates and cannot be motivated logically. Nevertheless, having introduced them, we find that in quantum mechanics there is an abelian gauge field of order $\hbar$ (or higher), that remains undetermined and vanishes in the classical limit $\hbar \to 0$.

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2 Structure of manifold with non-transitive group of isometries

Let us consider a p-dimensional Riemannian manifold M equipped with the metric $\{\eta_{MN} : M, N = \overline{1,p}\}$ in which the action of the n-dimensional intransitive group of isometries G is given ($n < p$). This means, according to [1], that the Killing equations

$$v^P \partial_P \eta_{MN} + \eta_{MP} \partial_N v^P + \eta_{PN} \partial_M v^P = 0 \tag{2.1}$$

have n independent solutions $\{v^N_A : N = \overline{1,p}; A = \overline{1,n}\}$.

The set of vector fields $\{v^M_A \partial_M : A = \overline{1,n}\}$ describes the representation of the Lie algebra Lie(G) of the Lie group G acting on M. Due to this fact each vector from this set obeys the equation

$$v^N_A \partial_N v^M_B - v^N_B \partial_N v^M_A = C^C{}_{AB} v^M_C, \tag{2.2}$$

where $C^C{}_{AB}$ are the structure constants of G. This equality provides the following equation with $m = p - n$ independent solutions:

$$v^N_A \partial_N \varphi(x) = 0. \tag{2.3}$$

Let us consider a new coordinate system

$$\overline{x}^N := (\varphi^\alpha(x), x^\mu), \tag{2.4}$$

$$N = \overline{1,p}, \quad \alpha = \overline{1,m}, \quad \mu = \overline{m+1,p},$$

where $\{\varphi^\alpha(x) : \alpha = \overline{1,m}\}$ denote independent solutions of (2.3). In these coordinates the Killing vectors take the form

$$(\overline{v}^M_A) = (v^N_A \partial_N \varphi^\alpha, v^\mu_A) \equiv (0, v^\mu_A). \tag{2.5}$$

Further we will assume that such a coordinate system, denoted by $\{x^N\}$, on M is given and all the geometrical objects are expressed in terms of it. The Killing vector can be rewritten as

$$v_i^M = \delta_\mu^M v_i^\mu, \tag{2.6}$$

where the group index $i = \overline{m+1,p}$ is used instead of $A = \overline{1,n}$ (the indices i, j, k, ... play the same roles as A, B, C, ...).

We introduce the inverse of the matrix $\{v_i^\mu : \mu = \overline{m+1,p}, i = \overline{m+1,p}\}$ by

$$e_\mu^i v_j^\mu = \delta_j^i, \quad e_\mu^i v_i^\nu = \delta_\mu^\nu. \tag{2.7}$$

Using (2.2), (2.6) and (2.7) one can prove that the matrix $\{e_\mu^i\}$ obeys the Maurer–Cartan equation

$$\partial_\mu e_\nu^i - \partial_\nu e_\mu^i = -C^i{}_{jk} e_\mu^j e_\nu^k. \tag{2.8}$$

It is easy to obtain the following equations by differentiating:

$$\begin{aligned} D_M v_i^\mu &= \partial_M v_i^\mu + \Lambda_{M\nu}^\mu v_i^\nu = 0, \\ D_M e_\mu^i &= \partial_M e_\mu^i - \Lambda_{M\mu}^\nu e_\nu^i = 0, \end{aligned} \tag{2.9}$$

where the object

$$\Lambda_{M\nu}^\mu = v_i^\mu \partial_M e_\nu^i \tag{2.10}$$

corresponds to the right group connection on M (see [1]).

Using (2.7) and (2.8) one can easily prove the following property of $\Lambda_{M\nu}^\mu$:

$$\Lambda_{\mu\nu}^\sigma - \Lambda_{\nu\mu}^\sigma = -C_{jk}^i v_i^\sigma e_\mu^j e_\nu^k. \tag{2.11}$$

Using the relations obtained above we can write down the functional features of the metric in the special coordinate system described by (2.4). To do this, we transform the Killing equation (2.1) into a new coordinate description:

$$\begin{aligned} &\partial_\mu \eta^{\alpha\beta} = 0, \\ &\partial_\mu \eta_{\alpha\nu} = \eta_{\alpha\sigma}\Lambda_{\nu\mu}^\sigma - \eta_{\nu\sigma}\Lambda_{\alpha\mu}^\sigma, \\ &\partial_\mu \eta_{\alpha\beta} = \eta_{\alpha\nu}\Lambda_{\beta\mu}^\nu + \eta_{\beta\nu}\Lambda_{\alpha\mu}^\nu. \end{aligned} \tag{2.12}$$

The matrix $\{\eta_{\mu\nu} : \mu, \nu = \overline{m+1,p}\}$ has the meaning of the metric on the orbit of the action of G on M, that is isomorphic to G.

Making the assumption that $\{\eta_{\mu\nu}\}$ is a non-degenerate matrix (so its inverse exists) we introduce the following objects:

$$A_\alpha^\mu = g^{\mu\nu}\eta_{\alpha\nu}, \quad g^{\mu\nu}\eta_{\nu\sigma} = \delta_\sigma^\mu \tag{2.13}$$

(here the matrix $\{g^{\mu\nu}\}$ is the inverse of $\{\eta_{\mu\nu}\}$). Then $\eta_{\alpha\mu} = g_{\mu\nu}A_\alpha^\nu$ and

$$D_\mu A_\alpha^\nu = \partial_\mu A_\alpha^\nu + \Lambda_{\mu\sigma}^\nu A_\alpha^\sigma = \Lambda_{\alpha\mu}^\nu, \tag{2.14}$$

as follows from (2.12).

Due to (2.13) and (2.14) the second equation in (2.12) is equivalent to

$$\partial_\mu(\eta_{\alpha\beta} - g_{\mu\nu}A_\alpha^\mu A_\beta^\nu) = 0. \tag{2.15}$$

Hence, the matrix

$$g_{\alpha\beta} = \eta_{\alpha\beta} - g_{\mu\nu}A_\alpha^\mu A_\beta^\nu \tag{2.16}$$

depends only on the coordinates $\{x^\alpha : \alpha = \overline{1,m}\}$ and is independent on $\{x^\mu : \mu = \overline{m+1,p}\}$. The metric tensor $\{\eta_{MN}\}$, describing the manifold M can be rewritten in terms of the objects $g_{\mu\nu}$, A_α^μ, $g_{\alpha\beta}$, as follows:

$$\{\eta_{MN}\} = \begin{pmatrix} g_{\alpha\beta} + g_{\rho\sigma}A_\alpha^\rho A_\beta^\sigma & g_{\mu\rho}A_\beta^\rho \\ A_\alpha^\rho g_{\rho\nu} & g_{\mu\nu} \end{pmatrix}, \tag{2.17}$$

$$\{\eta^{MN}\} = \begin{pmatrix} g^{\alpha\beta} & -A_\gamma^\mu g^{\gamma\beta} \\ -g^{\alpha\gamma}A_\beta^\nu & g^{\mu\nu} + g^{\gamma\delta}A_\gamma^\mu A_\delta^\nu \end{pmatrix}. \tag{2.18}$$

This decomposition is similar to the Kaluza–Klein one. In (2.18) $\{g^{\alpha\beta}\}$ denotes the inverse of $\{g_{\alpha\beta}\}$. If $\{\eta_{MN}\}$ and $\{g_{\mu\nu}\}$ are non-degenerate matrices, the matrix $\{g^{\alpha\beta}\}$ exists due to the following property:

$$\det\{\eta_{MN}\} = \det\{g_{\alpha\beta}\} \cdot \det\{g_{\mu\nu}\}.$$

The functional properties of the metric (2.17) are determined by the following equations:

$$\begin{cases} \partial_\mu g_{\alpha\beta} = 0, \\ \partial_\rho g_{\mu\nu} = g_{\mu\sigma}\Lambda_{\rho\mu}^\sigma + g_{\nu\sigma}\Lambda_{\rho\nu}^\sigma, \\ \partial_\mu A_\alpha^\nu = \Lambda_{\alpha\mu}^\nu - A_\alpha^\rho \Lambda_{\rho\mu}^\nu. \end{cases} \tag{2.19}$$

By their form as written, the matrices (2.17), (2.18) correspond to the analogous ones in [3], but their meaning is different.

From the geometrical point of view the construction we are considering corresponds to the principal bundle with a total space M, a base space M/G and a structure group G. The coordinates $\{x^\alpha\}$ from the set $\{x^M\} = \{x^\alpha, x^\mu\}$ describes the local coordinate system in M/G and the matrix $\{g_{\alpha\beta}\}$ the metric tensor of M/G.

The projection map of the principal bundle (M/G) (M, G) has the form

$$\begin{aligned} p : M &\longrightarrow M/G, \\ \{x^\alpha, x^\mu\} &\longrightarrow \{x^\alpha\}. \end{aligned}$$

The vector fields

$$\hat{D}_i = v_i^\mu \partial_\mu, \quad \hat{D}_\alpha = \partial_\alpha - A_\alpha^\mu \partial_\mu \tag{2.20}$$

form the bases of vertical and horizontal fields, respectively. One can observe this from

$$\mathrm{d}p(\hat{D}_i|_{\{x^\alpha,x^\mu\}}) = 0, \quad \mathrm{d}p(\hat{D}_\alpha|_{\{x^\alpha,x^\mu\}}) = \partial_\alpha|_{\{x^\alpha\}}. \tag{2.21}$$

The objects $\{A_\alpha^\mu\}$ defined above correspond to the Lie(G) valued connection 1-form on M

$$\omega^i = e_\mu^i \left(\mathrm{d}x^\mu + A_\alpha^\mu \mathrm{d}x^\alpha\right). \tag{2.22}$$

Now we write down some useful relations for the Riemannian geometry on M, which describe its features explicitly.

Let ∇_α be a standard covariant derivative constructed with the metric $\{g_{\alpha\beta}\}$. Define the operator on M/G that generalizes the operator $\hat{\nabla}_\alpha$ by

$$\hat{\nabla}_\alpha B_\beta = \hat{D}_\alpha B_\beta - \Gamma^\gamma{}_{\alpha\beta} B_\gamma \tag{2.23}$$

for some vector field B_α on M/G depending on coordinates $\{x^\mu\}$. Then

$$[\hat{\nabla}_\alpha, \hat{\nabla}_\beta] B_\gamma = [\hat{D}_\alpha, \hat{D}_\beta] B_\gamma - R^\delta{}_{\alpha\beta\gamma} B_\delta, \tag{2.24}$$

where $R^\delta{}_{\alpha\beta\gamma}$ is the curvature tensor for the metric $g_{\alpha\beta}$.

The explicit expression for the commutator $[\hat{D}_\alpha, \hat{D}_\beta]$ can be obtained from its action on a scalar function $f(x^\alpha, x^\mu)$. Performing a simple calculation we can observe that

$$\begin{aligned}[\hat{D}_\alpha, \hat{D}_\beta] f = &- (\partial_\alpha A^\mu_\beta - \partial_\beta A^\mu_\alpha)\partial_\mu f \\ &+ (A^\mu_\alpha \partial_\mu A^\nu_\beta - A^\mu_\beta \partial_\mu A^\nu_\alpha)\partial_\nu f.\end{aligned} \tag{2.25}$$

Using (2.19) in (2.26)

$$\begin{aligned}A^\mu_\alpha \partial_\mu A^\nu_\beta - A^\mu_\beta \partial_\mu A^\nu_\alpha = &-C^i{}_{ik} A^j_\alpha A^k_\beta v^\nu_i \\ &+ (\partial_\alpha A^\nu_\beta - \partial_\beta A^\nu_\alpha) + (\partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha) v^\nu_i,\end{aligned} \tag{2.26}$$

where

$$A^i_\alpha = e^i_\mu A^\mu_\alpha, \quad \partial_\mu A^i_\alpha = -C^i{}_{jk} e^j_\mu A^k_\alpha + \partial_\alpha e^i_\mu. \tag{2.27}$$

Due to (2.27) and (2.27) the expression (2.26) can be rewritten as

$$[\hat{D}_\alpha, \hat{D}_\beta] f = -F^i_{\alpha\beta} \hat{D}_i f, \quad \hat{D}_i = v^\mu_i \partial_\mu. \tag{2.28}$$

We have

$$F^i_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha + C^i{}_{jk} A^j_\alpha A^k_\beta. \tag{2.29}$$

Observing (2.28) and (2.29) we can draw the conclusion that the objects $\{A^i_\alpha\}$ can be interpreted as gauge fields defined on M/G with the strength tensor $F^i_{\alpha\beta}$.

The other relations between the basic fields have the form

$$[\hat{D}_i, \hat{D}_j] = C^k{}_{ij} \hat{D}_k, \quad [\hat{D}_\alpha, \hat{D}_i] = 0.$$

At the end of this section we consider classes of coordinate transformations $x^\mu \to \overline{x}^\mu$ preserving the form of the metric (2.17) and (2.18) and its functional structure. Evidently, one such class consists of coordinate transformations on the orbit of the action of G:

$$\begin{cases} \overline{x}^\alpha = x^\alpha, \\ \overline{x}^\mu = \overline{x}^\mu(x^\nu). \end{cases} \tag{2.30}$$

Under such a transformation $g_{\mu\nu}$ and A^μ_α behave as a tensor and a covariant vector, respectively (the geometric properties are described by μ, ν, ...). At the same time $g_{\alpha\beta}$ transforms as a scalar.

The coordinate change on M/G

$$\begin{cases} \overline{x}^\alpha = \overline{x}^\alpha(x^\beta), \\ \overline{x}^\mu = x^\mu \end{cases} \tag{2.31}$$

is of the same class.

The objects $g_{\alpha\beta}$, A^μ_α transform as a tensor and a contravariant vector respectively (the geometric properties are described by α, β, ...), while $g_{\alpha\beta}$ transforms as a scalar.

The third class of transformations is described by

$$\begin{cases} \overline{x^\alpha} = x^\alpha, \\ \overline{x}^\mu = x^\mu + \varphi^\mu(x^\alpha). \end{cases} \tag{2.32}$$

The main geometrical objects transform under (2.32) as

$$\begin{aligned}&\overline{g}_{\mu\nu}(\overline{x}) = g_{\mu\nu}(x), \quad \overline{g}_{\alpha\beta}(\overline{x}) = g_{\alpha\beta}(x), \\ &\overline{A}^\mu_\alpha(\overline{x}) = A^\mu_\alpha - \partial_\alpha \varphi^\mu(x^\alpha).\end{aligned}$$

An arbitrary vector F_M on M can be decomposed into two objects that are invariant under the transformation (2.32):

$$\mathcal{F}_\alpha = F_\alpha - A^\mu_\alpha F_\mu, \quad \mathcal{F}_\mu = F_\mu.$$

This decomposition means the extraction of the horizontal part of F_M.

3 Lagrangian and variational principle

Constructing quantum mechanics on a Riemannian manifold M with the non-transitive group of isometries G in terms of a variational principle, as in our previous papers, we assume that the coordinate operators x^M form a complete set of commuting observables.

The quantum Lagrangian can be written as

$$L = \frac{1}{2} \dot{x}^M \eta_{MN}(x) \dot{x}^N - U_q(x), \tag{3.1}$$

where $U_q(x)$ is some function that provides the scalar transformation law of L under a general non-degenerate coordinate transformation $x \to \overline{x} = \overline{x}(x)$ (see [2,3]). As has been pointed out in [2], its explicit expression can be determined in the case when the commutator $[x^M, \dot{x}^N]$ is a function of only $\{x^M\}$. This is a standard assumption in the formulation of quantum mechanics on a Riemannian manifold (see [4] and the motivation in [2]). The function U_q appears to have values of order $\hbar^2$.

In the special coordinate system, introduced in (2.4), the metric tensor $\{\eta_{MN}\}$ receives a form that is similar to the one appearing in Kaluza–Klein theories. The Lagrangian can be written as

$$\begin{aligned}L = &\frac{1}{2} \left(\dot{x}^\mu + \dot{x}^\alpha A^\mu_\alpha\right) g_{\mu\nu} \left(\dot{x}^\nu + A^\nu_\beta \dot{x}^\beta\right) \\ &+ \frac{1}{2} \dot{x}^\alpha g_{\alpha\beta} \dot{x}^\beta - U_q(x).\end{aligned} \tag{3.2}$$

The Euler–Lagrange dynamical equations, according to [2], can be derived from the action principle in the form of the variational equation $\delta L = 0$, that corresponds to the infinitesimal coordinate variation $x^M \to x^M + \delta x^M(x)$.

As has been shown in [2], the equality $\delta L = 0$ holds if and only if the variations δx^M are Killing vectors. In the present case, in contradiction to the one investigated in [2,3], the dimension of the manifold M is higher than the number of linear independent Killing vectors $\{v_A^M : M = \overline{1,p}, A = \overline{1,n}\}$ (i.e. $n < p$).

Extracting the total time derivative, as in [2], we can rewrite the Lagrangian as

$$\begin{aligned}\delta L = &\frac{\mathrm{d}}{\mathrm{d}t}\left(p_M \circ \delta x^M\right) - \dot{p}_M \circ \delta x^M \\ &+\frac{1}{2}\dot{x}^M\left(\delta x^P \partial_P \eta_{MN}\right)\dot{x}^N - \delta x^M \partial_M U_q \\ &+\frac{1}{2}\left[\delta x^M, \frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x}^N, \eta_{MN}\right]\right], \end{aligned} \tag{3.3}$$

where $p_M = \eta_{MN} \circ \dot{x}^N$ is for the momentum operator on M.

In accordance with [2], the object

$$G = p_M \circ \delta x^M \tag{3.4}$$

has the meaning of the generator of permissible variations.

Further we decompose the variation δx^M in terms of the basis of linear independent Killing vectors (2.6):

$$\delta x^M = v_i^M \varepsilon^i, \quad \varepsilon^i = \mathrm{const}. \tag{3.5}$$

This allows us to rewrite (3.4) as

$$G = \left(p_M \circ v_i^M\right)\varepsilon_i = p_i \varepsilon^i, \quad p_i := p_M \circ v_i^M. \tag{3.6}$$

In the special coordinate system (2.4) the objects $\{p_i\}$ have the form

$$\begin{aligned} &p_i = p_\mu \circ v_i^\mu, \\ &p_\mu = \eta_{\mu M} \circ \dot{x}^M = g_{\mu\nu} \circ \left(\dot{x}^\nu + A_\alpha^\nu \circ \dot{x}^\alpha\right). \end{aligned} \tag{3.7}$$

4 Algebra of commutation relations

Following the procedure presented in [2], we derive the commutation relations for quantum theory on M performing an investigation of the properties of the permissible variations connected with the action of the group G of isometries on M. From the definition of permissible variations in terms of the representation of G on M (in the special coordinate system) we have

$$\delta_i x^\mu = v_i^\mu = \frac{1}{\mathrm{i}\hbar}\left[x^\mu, p_i\right], \tag{4.1}$$

$$\delta_i x^\alpha = 0 = \frac{1}{\mathrm{i}\hbar}\left[x^\alpha, p_i\right]. \tag{4.2}$$

(Here $\delta x^\mu = \varepsilon^i \delta_i x^\mu$.) Using the assumption about the commutativity of the coordinates $\{x^M\}$ and taking into account the fact that the matrix $\{v_i^\mu\}$ is non-degenerate (i.e. $\det v_i^\mu \neq 0$), we can conclude from (4.1) and (4.2) that

$$[x^\mu, p_\nu] = \mathrm{i}\hbar\delta_\nu^\mu, \quad [x^\alpha, p_\mu] = 0. \tag{4.3}$$

The variation of the arbitrary function f which depends on the coordinates $\{x^M\}$ can be defined as follows:

$$\delta_i f(x) = v_i^\mu \partial_\mu f(x) = \frac{1}{\mathrm{i}\hbar}[f, p_\mu]. \tag{4.4}$$

Hence

$$[f(x), p_\mu] = \mathrm{i}\hbar\partial_\mu f(x). \tag{4.5}$$

The variations of the velocity operators are

$$\delta_i \dot{x}^\mu = \frac{\mathrm{d}v_i^\mu}{\mathrm{d}t} = \dot{x}^M \circ \partial_M v_i^\mu = \frac{1}{\mathrm{i}\hbar}[\dot{x}^\mu, p_i], \tag{4.6}$$

$$\delta_i \dot{x}^\alpha = \frac{\mathrm{d}v_i^\alpha}{\mathrm{d}t} = 0 = \frac{1}{\mathrm{i}\hbar}[\dot{x}^\alpha, p_i]. \tag{4.7}$$

Using (3.6) and (3.7) we can express the velocity operators $\{\dot{x}^\mu\}$ in (4.6) in terms of the momentum operators $\{p_\mu\}$. Finally, we have

$$[p_\mu, p_i] = -\mathrm{i}\hbar p_\nu \circ \partial_\mu v_i^\nu. \tag{4.8}$$

The commutator (4.8) can be presented in another form, taking into account (2.2):

$$[p_i, p_j] = -\mathrm{i}\hbar C^i{}_{ik} p_k. \tag{4.9}$$

Similarly, due to (2.2) and (4.5), we can write

$$[p_\mu, p_\nu] = 0. \tag{4.10}$$

The commutators (4.3), (4.5), (4.9) and (4.10) completely define quantum mechanics on the orbit of the action of G on M analyzed in [2].

Further, returning to the manifold M/G, we define the momentum operator

$$p_\alpha = \eta_{\alpha M} \circ \dot{x}^M \tag{4.11}$$

and the auxiliary operator

$$\pi_\alpha = g_{\alpha\beta} \circ \dot{x}^\beta = p_\alpha - A_\alpha^\mu \circ p_\mu. \tag{4.12}$$

Due to $\partial_\mu g_{\alpha\beta} = 0$, the objects p_μ and π_μ are connected by the following commutation relations:

$$[p_\alpha, p_i] = -\mathrm{i}\hbar \circ \partial_\alpha v_i^\mu, \quad [\pi_\alpha, p_i] = 0. \tag{4.13}$$

These are all the commutators which can be obtained immediately from the generator of permissible variations (3.6). Note, that the commutation relations presented above are form invariant under a general coordinate transformation $x^M \to \overline{x}^M = \overline{x}^M(x)$. The derivation of the remaining commutators requires the usage of operator equalities and introducing some additional assumptions.

It follows from the basic assumptions that the commutator between coordinate and momentum operators on M is a function of only $\{x^M\}$, i.e.

$$[x^M, p_N] = \mathrm{i}\hbar B^M_N(x). \tag{4.14}$$

Due to the commutation relations obtained above the parts of the matrix B are already defined, namely $B^M_\nu = \delta^M_\nu$. Then we can write

$$B = \begin{pmatrix} B^\alpha_\beta & B^\mu_\beta \\ 0 & \delta^\mu_\nu \end{pmatrix}, \tag{4.15}$$

where the unknown functions $B^\alpha_\beta, B^\mu_\beta$ can be written in the following form motivated by the correspondence principle:

$$B^\alpha_\beta = \delta^\alpha_\beta + b^\alpha_\beta, \quad B^\mu_\alpha = b^\mu_\alpha. \tag{4.16}$$

The new unknown objects b^α_β and b^μ_α in (4.15) are functions of x of order $\hbar^2$ (or higher). To derive their operator properties we can use the commutator of the structure equation

$$v^\mu_i \partial_\mu v^\nu_j - v^\mu_j \partial_\mu v^\nu_i = C^k{}_{ij} v^\nu_k, \tag{4.17}$$

with the momentum operator p_α. Using the relation

$$[v^\mu_i, p_j] = \mathrm{i}\hbar \hat{D}_j v^\mu_i \tag{4.18}$$

(the "long derivative" $\hat{D}_i$ was introduced in (2.20) and (2.21)) we can rewrite (4.17) as

$$[v^\mu_i, p_j] - [v^\mu_j, p_i] = -C^k{}_{ij} v^\mu_k. \tag{4.19}$$

Therefore, using the Jacobi identity, we arrive at the following operator equality:

$$[[v^\mu_i, p_j], p_\alpha] = -\hbar^2 \partial_\nu v^\mu_i \partial_\alpha v^\nu_j + \mathrm{i}\hbar \hat{D}_j [v^\mu_i, p_\alpha]. \tag{4.20}$$

Further, taking the antisymmetrization of (4.20) with respect to the indices i, j we can find the equation

$$\hat{D}_i \varphi^\mu{}_{j\alpha} - \hat{D}_j \varphi^\mu{}_{i\alpha} = C^k{}_{ij} \varphi^\mu_{k\alpha}, \tag{4.21}$$

where we introduce the unknown function of coordinates defined by

$$\varphi^\mu{}_{i\alpha} = \frac{1}{\mathrm{i}\hbar}[v^\mu_i, p_\alpha] - \partial_\alpha v^\mu_i. \tag{4.22}$$

Taking into account the structure of the operator $\hat{D}_i$ we can conclude that the solution of (4.21) has the form

$$\varphi^\mu{}_{i\alpha} = \hat{D}_i \varphi^\mu_\alpha, \tag{4.23}$$

where φ^μ_α is a new unknown function of $\{x^M\}$.

Making the substitution $p_i \equiv v^\mu_i \circ p_\mu$ in (4.13) we obtain

$$[p_\alpha, p_\mu] = \mathrm{i}\hbar \partial_\mu \varphi^\nu_\alpha \circ p_j. \tag{4.24}$$

Now we are going to show the relation between the objects φ^μ_α and B^ν_α. To do this let us write down the Jacobi identities including the coordinate operators x^M and momentum operators p_α, p_μ:

$$\frac{1}{\mathrm{i}\hbar}[x^\alpha, [p_\beta, p_\mu]] \equiv 0$$
$$= \frac{1}{\mathrm{i}\hbar}([[x^\alpha, p_\beta], p_\mu] - [[x^\alpha, p_\mu], p_\beta]) = \mathrm{i}\hbar \partial_\mu B^\alpha_\beta, \tag{4.25}$$
$$\frac{1}{\mathrm{i}\hbar}[x^\mu, [p_\alpha, p_\nu]] \equiv \mathrm{i}\hbar \partial_\nu \varphi^\mu_\alpha$$
$$= \frac{1}{\mathrm{i}\hbar}([[x^\mu, p_\alpha], p_\nu] - [[x^\mu, p_\nu], p_\alpha]) = \mathrm{i}\hbar \partial_\nu B^\mu_\alpha. \tag{4.26}$$

From (4.25) we can draw the conclusion of the independence of B^α_β (and, in consequence, b^α_β) on x^μ, because

$$\partial_\mu B^\alpha_\beta = \partial_\mu b^\alpha_\beta = 0. \tag{4.27}$$

At the same time, it follows from (4.26) that the differential connection between φ^μ_α and B^μ_α (and, in consequence, b^α_β) is

$$\partial_\mu \varphi^\nu_\alpha = \partial_\mu B^\nu_\alpha = \partial_\mu B^\nu_\alpha. \tag{4.28}$$

The object φ^μ_α appears in all the formulae we deal with under the derivative operator ∂_μ. Hence, using (4.28) we can identify φ^μ_α with b^μ_α, i.e. we take

$$\varphi^\mu_\alpha(x) = b^\mu_\alpha(x). \tag{4.29}$$

Due to the structure equation the commutator for v^μ_i and p_α reads

$$\frac{1}{\mathrm{i}\hbar}[v^\mu_i, p_\alpha] = \partial_\alpha v^\mu_i + v^\nu_i \partial_\nu b^\mu_\alpha. \tag{4.30}$$

On the other hand, in accordance with (4.14), this commutator is equal to

$$\frac{1}{\mathrm{i}\hbar}[v^\mu_i p_\alpha] = B^M_\alpha \partial_M v^\mu_i = \partial_\alpha v^\mu_i + b^M_\alpha \partial_M v^\mu_i. \tag{4.31}$$

By comparing (4.30) and (4.31) we obtain the equation for the objects $\{b^M_\alpha\}$:

$$v^M_i \partial_M b^\mu_\alpha - b^M_\alpha v^\mu_i = 0, \quad \partial_\mu b^\alpha_\beta = 0. \tag{4.32}$$

This equation has a transparent geometrical meaning. In accordance with [1], the object $\{b^M_\alpha\}$ determines vector fields on M that generate the one-parametric group of coordinate transformations commuting with isometries (isometric transformations). In the general case these transformations do not form the representation of an $p - n$-dimensional group.

As the solutions of (4.32), the objects are not determined unequivocally. If b^M_α denotes the solution of (4.32), its linear combination

$$\bar{b}^M_\alpha(x) = \varepsilon^\beta_\alpha(x^\gamma) b^M_\beta(x) \tag{4.33}$$

also obeys (4.32) for arbitrary functions $\varepsilon^M_\alpha = \varepsilon^M_\alpha(x^\gamma)$.

Therefore, the remaining commutation relations are

$$\begin{aligned} &[x^\alpha, p_\beta] = \mathrm{i}\hbar\left(\delta^\alpha_\beta + b^\alpha_\beta\right), \\ &[x^\mu, p_\beta] = \mathrm{i}\hbar b^\mu_\beta, \\ &[p_\mu, p_\alpha] = -\mathrm{i}\hbar\partial_\mu b^\nu_\alpha \circ p_\nu, \end{aligned} \quad (4.34)$$

where b^M_α obeys (4.32).

In order to construct the self-contained algebra of the commutation relations for quantum mechanics on M we have to obtain the explicit form of $[p_\alpha, p_\beta]$. In the general case we can write

$$[p_\alpha, p_\beta] = \mathrm{i}\hbar\left({F^M}_{\alpha\beta} \circ p_M + \varphi_{\alpha\beta}\right), \quad (4.35)$$

where ${F^M}_{\alpha\beta}$, $\varphi_{\alpha\beta}$ are unknown tensors of the coordinates on M. To determine these objects we use the Jacobi identities for the operators x^α, p_β, p_γ and x^μ, p_α, p_β. After a simple calculation we find that

$${F^\gamma}_{\alpha\beta} B^\delta_\gamma = -\left(B^\gamma_\alpha \partial_\gamma B^\delta_\beta - B^\gamma_\beta \partial_\gamma B^\delta_\alpha\right), \quad (4.36)$$

$${F^\mu}_{\alpha\beta} + B^\mu_\gamma F^\gamma_{\alpha\beta} = -\left(B^M_\alpha \partial_M B^\mu_\beta - B^M_\beta \partial_M B^\mu_\alpha\right). \quad (4.37)$$

These equations allow us to define $F^M_{\alpha\beta}$ as a function of B^M_α.

To investigate the functional properties of the object $\varphi_{\alpha\beta}(x)$ it is convenient to introduce the following operators:

$$\overline{p}_\alpha = p_\alpha - b^\mu_\alpha \circ p_\mu. \quad (4.38)$$

According to (4.35), the operators (4.38) satisfy the following commutation relations:

$$[\overline{p}_\alpha, p_\mu] = 0, \quad [\overline{p}_\alpha, \overline{p}_\beta] = \mathrm{i}\hbar\left({F^\gamma}_{\alpha\beta} \circ \overline{p}_\gamma + \varphi_{\alpha\beta}\right). \quad (4.39)$$

From the relations

$$[p_\mu, [\overline{p}_\alpha, \overline{p}_\beta]] = 0, \quad \partial_\mu {F^\gamma}_{\alpha\beta} = 0, \quad (4.40)$$

obtained from (4.38) and (4.39) we can conclude that

$$\partial_\mu \varphi_{\alpha\beta} = 0, \quad (4.41)$$

i.e. $\varphi_{\alpha\beta}$ are functions of only the coordinates on the quotient space M/G.

The further development of the theory can be performed only due to additional assumptions about the structure of the matrix B^M_N, that cannot be described in terms of the basic principles, having made a base of our quantization scheme.

Let B^M_N have the simplest form, $B^M_N = \delta^M_N$. In this case, as follows from results obtained above, the quantum mechanics is described by

$$[x^M, p_N] = \mathrm{i}\hbar\delta^M_N, \quad [x^M, \dot{x}^N] = \mathrm{i}\hbar\eta^{MN}. \quad (4.42)$$

The relations (4.42) can be divided into two parts. The first one corresponds to quantum mechanics on the orbit,

$$[x^\mu, p_\nu] = \mathrm{i}\hbar\delta^\mu_\nu, \quad [x^\mu, x^\nu] = 0, \quad [p_\mu, p_\nu] = 0, \quad (4.43)$$

while the second one describes the quantum mechanics on the quotient space M/G:

$$\begin{aligned} &[x^\alpha, x^\beta] = 0, \qquad [x^\alpha, \pi_\beta] = \mathrm{i}\hbar\delta^\alpha_\beta, \\ &[\pi_\alpha, \pi_\beta] = \mathrm{i}\hbar\left(p_\mu \circ {F^\mu}_{\alpha\beta} + \varphi_{\alpha\beta}\right). \end{aligned} \quad (4.44)$$

The remaining commutation relations contained in (4.42) have the form

$$\begin{aligned} &[x^\alpha, x^\mu] = 0, \qquad [\pi_\alpha, p_\mu] = -\mathrm{i}\hbar\partial_\mu A^\nu_\alpha \circ p_\nu, \\ &[x^\mu, \pi_\alpha] = -\mathrm{i}\hbar A^\mu_\alpha, \quad [x^\alpha, p_\mu] = 0. \end{aligned} \quad (4.45)$$

It is essential that the features of quantum mechanics on M/G depend on the tensor $\varphi_{\alpha\beta}$, which cannot be determined in our quantization scheme. This object can be viewed as the strength tensor of some abelian gauge field. To show this, we have to take into account that the commutator (4.35) can be rewritten in a simpler form due to the restriction $B^M_N = \delta^M_N$, namely

$$[p_\alpha, p_\beta] = \mathrm{i}\hbar\varphi_{\alpha\beta}(x^\gamma). \quad (4.46)$$

Using (4.46) we find from the Jacobi identities for the operators p_α, p_β, p_γ that $\varphi_{\alpha\beta}$ obeys the relation

$$\partial_\alpha\varphi_{\beta\gamma} + \partial_\beta\varphi_{\gamma\alpha} + \partial_\gamma\varphi_{\alpha\beta} = 0. \quad (4.47)$$

Then the evident solution of (4.47) is

$$\varphi_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \quad (4.48)$$

where A_α is some unknown abelian gauge field.

5 Quantum Lagrangian on M

Following [2] we introduce the following operator as the Lagrangian of a point particle with unit mass:

$$L(x, \dot{x}) := \frac{1}{2}(\dot{x}, \dot{x}) = \frac{1}{2}(p, p). \quad (5.1)$$

Here $(\cdot,\cdot)$ means the scalar product on M that is invariant under a general coordinate transformation $x \to \overline{x}(x)$. Its construction has been discussed in detail in [2]. In the present paper we modify this definition for the case

$$[x^M, p_N] = \mathrm{i}\hbar B^M_N(x), \quad (5.2)$$

where $B^M_N \neq \delta^M_N$. One can simply show that B^M_N is a tensor field on M (and, consequently, $b^M_N = B^M_N - \delta^M_N$ is a tensor field too).

Under a general coordinate transformation $x \to \overline{x}(x)$ on M the momentum operator transforms as

$$\begin{aligned} p_M \to \overline{a}^N_M \circ p_N &= \overline{a}^N_M p_N + \frac{1}{2}[p_N, \overline{a}^N_M] \\ &= \overline{a}^N_M p_N - \frac{\mathrm{i}\hbar}{2}(\partial_N \overline{a}^N_M + b^P_M \partial_P \overline{a}^N_M), \end{aligned} \quad (5.3)$$

$$\begin{aligned} p_M \to \overline{a}^N_M \circ p_N &= p_N \overline{a}^N_M - \frac{1}{2}[p_N, \overline{a}^N_M] \\ &= p_N \overline{a}^N_M + \frac{\mathrm{i}\hbar}{2}(\partial_N \overline{a}^N_M + b^P_M \partial_P \overline{a}^N_M). \end{aligned} \quad (5.4)$$

Taking into account the transformation law of the Christoffel symbols Γ^P_{MN}, constructed with the metric η_{MN}, we can write down the derivatives of the transformation matrices:

$$\partial_N \overline{a}^N_M = \overline{\Gamma}_M + \overline{a}^N_M \Gamma_N, \quad \Gamma_N = \Gamma^M_{MN}, \tag{5.5}$$

$$\partial_N \overline{a}^P_M = \overline{\Gamma}^S_{LM} \overline{a}^L_N \overline{a}^P_S - \overline{a}^S_M \Gamma^P_{NS}. \tag{5.6}$$

Let us introduce the notation

$$W_M = \Gamma^L_{MN} b^N_{\mathrm{L}}. \tag{5.7}$$

This object transforms under a general coordinate transformation on M as

$$\overline{W}_M = \overline{a}^N_M W_N + b^N_{\mathrm{L}} \partial_N \overline{a}^L_M. \tag{5.8}$$

Hence, taking into account the transformation laws of p_M, Γ_M and W_M, we can define the following "left" and "right" parts of the momentum operator:

$$\pi_M = p_M - \frac{\mathrm{i}\hbar}{2}(\Gamma_M + W_M),$$
$$\pi^\dagger_M = p_M + \frac{\mathrm{i}\hbar}{2}(\Gamma_M + W_M), \tag{5.9}$$

that transform as

$$\pi^\dagger_M \to \overline{\pi}^\dagger_M = \overline{a}^N_M \pi^\dagger_N, \quad \pi_M \to \overline{\pi}_M = \pi_N \overline{a}^N_M.$$

Following [2] we define the scalar norm of the momentum operator p_M on M in terms of the operators $\pi^\dagger_M$ and π_M by

$$(p,p) = \pi_M \eta^{MN} \pi^\dagger_N. \tag{5.10}$$

Using the result obtained above, the Lagrangian can be written in the form

$$L = \frac{1}{2}(\pi_\alpha - \pi_\mu A^\mu_\alpha) g^{\alpha\beta} (\pi^\dagger_\beta - A^\mu_\beta \pi^\dagger_\beta) + \frac{1}{2}\pi_\mu g^{\mu\nu} \pi^\dagger_\nu. \tag{5.11}$$

In the simplest case $B^M_N = \delta^M_N$ we can write it as

$$L = \frac{1}{2}\left(p_M - \frac{\mathrm{i}\hbar}{2}\Gamma_M\right)\eta^{MN}\left(p_M + \frac{\mathrm{i}\hbar}{2}\Gamma_M\right)$$
$$= \frac{1}{2}p_M \eta^{MN} p_N + \frac{\hbar^2}{4}\left(\partial_M \Gamma^M + \frac{1}{2}\Gamma_M \Gamma^M\right), \tag{5.12}$$

where

$$\Gamma_M = \Gamma^N_{MN}, \quad \Gamma^M = \eta^{MN}\Gamma_N. \tag{5.13}$$

The further analyses will be performed with the restriction $B^M_N = \delta^M_N$. Define the metric on the group manifold G:

$$\eta_{ij} = g_{\mu\nu} v^\mu_i v^\nu_j, \quad \partial_M \eta_{ij} = 0. \tag{5.14}$$

The norm of the momentum operator on G,

$$\mathcal{P}|_G = \{v^\mu_i \circ p_\mu : i = \overline{m = 1, p}\},$$

is defined as

$$(\mathcal{P}|_G, \mathcal{P}|_G) = p_i \eta^{ij} p_j. \tag{5.15}$$

From the Killing equations in the form $\nabla_M v^M = 0$ it follows that

$$\partial_M v^M_A \equiv \partial_\mu v^\mu_A = -\Gamma_\mu v^\mu_A. \tag{5.16}$$

At the same time we conclude from (2.10) that

$$\Gamma_\mu = -e^i_\mu \partial_\nu v^\nu_i = v^\nu_i \partial_\nu e^i_\mu = \Lambda^\nu{}_{\nu\mu}. \tag{5.17}$$

Performing a direct calculation, one can obtain

$$\Gamma_\alpha = \Gamma^M_{N\alpha} = \frac{1}{2}g^{\beta\gamma}\partial_\alpha g_{\beta\gamma} + \frac{1}{2}g^{\mu\nu}\partial_\alpha g_{\mu\nu}. \tag{5.18}$$

Finally, due to (5.17) and (5.18) the definition (5.15) can be written as

$$(\mathcal{P}|_G, \mathcal{P}|_G) = \left(p_\mu - \frac{\mathrm{i}\hbar}{2}\Gamma_\mu\right) g^{\mu\nu} \left(p_\mu + \frac{\mathrm{i}\hbar}{2}\Gamma_\mu\right), \tag{5.19}$$

i.e. the norm of a group momentum coincides with the double Lagrangian for a free particle on the orbit of G on M.

Using (5.9) we can transform the Lagrangian into another form:

$$L = \frac{1}{2}p_j g^{ij} p_j + \frac{1}{2}\left(p_\alpha - p_\mu A^\mu_\alpha - \frac{\mathrm{i}\hbar}{2}\hat{\Gamma}_\alpha\right)$$
$$\times g^{\alpha\beta}\left(p_\alpha - A^\mu_\alpha p_\mu + \frac{\mathrm{i}\hbar}{2}\hat{\Gamma}_\alpha\right), \tag{5.20}$$

where

$$\hat{\Gamma}_\alpha = \Gamma_\alpha - A^\mu_\alpha \Gamma_\mu.$$

So we can rewrite (5.19) in terms of $\pi_\alpha = p_\alpha - A^\mu_\alpha \circ p_\mu$ as

$$L = \frac{1}{2}p_i \eta^{ij} p_j + \frac{1}{2}\left(\pi_\alpha - \frac{\mathrm{i}\hbar}{2}\Omega_\alpha\right) g^{\alpha\beta}\left(\pi_\alpha + \frac{\mathrm{i}\hbar}{2}\Omega_\alpha\right), \tag{5.21}$$

where

$$\Omega_\alpha = \hat{\Gamma}_\alpha - \partial_\mu A^\mu_\alpha.$$

In order to simplify (5.21) we have to present the object Ω_α in the explicit form. Using the structure equation we obtain

$$\partial_\mu A^\mu_\alpha = -A^\nu_\alpha \Lambda^\mu_{\nu\mu} + \Lambda^\mu_{\alpha\mu}, \tag{5.22}$$

and, consequently,

$$\Omega_\alpha = \Gamma_\alpha + A^\mu_\alpha \Lambda^\nu_{[\mu\nu]} - \Lambda^\mu_{\alpha\mu}$$
$$= \Gamma_\alpha + v^\mu_i \partial_\alpha e^i_\mu + C^i{}_{ij} A^j_\alpha, \tag{5.23}$$

where

$$A^i_\alpha = e^i_\mu A^\mu_\alpha.$$

Further, taking into account (2.18) and the equality

$$\det\{\eta_{AB}\} = \det\{g_{\alpha\beta}\} \cdot \det\{v^\mu_i\}, \tag{5.24}$$

we arrive at the final expression for Ω_α:

$$\Omega_\alpha = \gamma_\alpha + \frac{1}{2}\eta^{ij}\partial_\alpha\eta_{ij} + C^i{}_{ij}A^j_\alpha, \qquad (5.25)$$

where $\gamma_\alpha = \gamma^\beta{}_{\alpha\beta}$, $\gamma^\alpha{}_{\beta\gamma}$ denotes the Christoffel symbol constructed with the metric $g_{\alpha\beta}$ of the quotient space.

The formulae (5.21) and (5.25) completely determine the final expression for the Lagrangian for a freely moving particle on M with the restriction $B^M_N = \delta^M_N$. The first term in (5.21) corresponds to quantum theory on the orbits of the action of G on M, while the second one describes the theory on the quotient space M/G. It is essential to point out here that the degrees of freedom $\{x^\mu\}$ determine the quantum mechanics on M/G by means of the last terms in (5.21).

6 Equations of motion for dynamics on Riemannian manifold

In this section we have the restriction of the function B^M_N to be equal to δ^M_N. The way of determining the dynamical equations of motion describing the particle moving on the manifold with an intransitive group of isometries differs from the one introduced in [2] only in details. From the condition $\delta L = 0$ of the permissible variations we find that

$$v^M_A \circ \left(\dot{p}_M + \frac{1}{2}p_{\mathrm{L}}\partial_M\eta^{LN}p_N + \partial_M U\right) = 0; \qquad (6.1)$$

here

$$U = \frac{\hbar^2}{4}\left(\partial_M\Gamma^M + \frac{1}{2}\Gamma_M\Gamma^M\right), \qquad (6.2)$$

where we have substituted the Lagrangian (5.12).

Using the condition $v^\mu_i e^i_\nu = 0$, where $\{e^i_\mu\}$ is the inverse of $\{v^\mu_i\}$ we obtain the equations of motion of the free particle for the degrees of freedom connected with the orbits of the action of G on M in the following form:

$$\dot{p}_\mu = -\frac{1}{2}p_M\partial_\mu\eta^{MN}p_N - \partial_\mu U. \qquad (6.3)$$

This equation is equivalent to the conservation law $\dot{p}_i = 0$ for the group momentum p^i.

As to the other degrees of freedom associated with M/G, its dynamical equations in Euler–Lagrange form must be equivalent to the ones in Heisenberg form:

$$\dot{p}_M = \frac{1}{\mathrm{i}\hbar}[p_M, H]. \qquad (6.4)$$

The explicit expression of the Hamiltonian can be obtained by comparison of (6.3) with (6.4) taking $M := \mu$. The Hamiltonian operator derived in such a way has the following form:

$$H = \frac{1}{2}p_M\eta^{MN}p_N + U + u_q(x^\alpha), \qquad (6.5)$$

where $u_q = u_q(x^\alpha)$ is some function that does not appear in (6.3) (because $\partial_\mu u_q \equiv 0$). The detailed calculation allows us to put $u_q = 0$ (see the remark below).

Hence, we can transform (6.1) into the following form:

$$\dot{p}_M = -\frac{1}{2}p_{\mathrm{L}}\partial_M\eta^{LN}p_N - \partial_M U - \left(\eta^{LN}\circ\varphi_{MN}\right)\circ p_{\mathrm{L}}, \qquad (6.6)$$

where

$$\varphi_{MN} = \frac{1}{\mathrm{i}\hbar}[p_M, p_N].$$

The restriction of M to μ reduces (6.6) to (6.3), as it must.

To prove the fact that (6.6) is correctly defined, we consider the operator H as the generator of time shifts $t \to \bar{t} = t + \delta t(t)$, where $\delta t(t)$ are permissible time variations. The result of the time variation of L must be in agreement with the equations of motion in Euler–Lagrange form, (6.6).

The time shift causes the following variations of coordinate and velocity operators:

$$\delta x = \overline{x}(\bar{t}) - x(t) = \dot{x}(t),$$
$$\delta\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\delta x - \dot{x}\frac{\mathrm{d}\delta t}{\mathrm{d}t}. \qquad (6.7)$$

Using the commutation relations we obtain

$$[x^M, \delta x^N] = \mathrm{i}\hbar\eta^{MN}\delta t, \quad [\dot{x}^M, \delta x^N] = [\dot{x}^M, \dot{x}^N]\delta t, \qquad (6.8)$$

where

$$[\dot{x}^M, \dot{x}^N] = \mathrm{i}\hbar\varphi^{MN} + \mathrm{i}\hbar f^{MN}_{\mathrm{L}} \circ \dot{x}^L, \qquad (6.9)$$

$$\varphi^{MN} = \eta^{ML}\eta^{NS}\varphi_{LS},$$
$$f^{MN}{}_P = \left(\eta^{MS}\eta^{NL} - \eta^{NS}\eta^{ML}\right)\partial_S\eta_{LP}. \qquad (6.10)$$

The variation of the Lagrangian caused only by the time shift appears in the variation of the action functional in the following combination:

$$\delta W = \int_{t_2}^{t_1} \delta_t L \mathrm{d}t, \quad \delta_t L = \delta L + L\frac{\mathrm{d}\delta t}{\mathrm{d}t}. \qquad (6.11)$$

The expression $\delta_t L$ reduces to

$$\delta_t L = \frac{\mathrm{d}(L\delta t)}{\mathrm{d}t} \qquad (6.12)$$

only due to the commutation relations.

On the other hand, extracting the total time derivative we obtain

$$\delta L = \frac{\mathrm{d}}{\mathrm{d}t}(H\delta t) + \frac{\mathrm{d}L}{\mathrm{d}t} - \left(\dot{p}_M - \frac{1}{2}p_N\partial_M\eta^{NL}p_{\mathrm{L}} - \partial_M U - p_N \circ \varphi^\mu{}_M\right)\circ\delta x^N; \qquad (6.13)$$

here H means the Hamiltonian. Hence, we can write

$$H = L, \quad \frac{\mathrm{d}H}{\mathrm{d}t} = 0, \qquad (6.14)$$

after comparing (6.12) with (6.13). Note that if we define the Hamiltonian with an additional term $u_q(x)$, we obtain after such a comparison that $u_q = 0$ (otherwise (6.12) and (6.13) do not agree). Therefore, we can see that (6.6) are consistent with the other conclusions.

The equations of motion for the degrees of freedom representing the orbits of the action of G on M are introduced above (see (6.3)). To write down the other dynamical equations describing the motion on M/G it is sufficient to assume $M \to \alpha \in \overline{1, m}$. In terms of π_α they are found in the form

$$\begin{aligned}\dot{\pi}_\alpha = &-\frac{1}{2}\pi_\beta \partial_\alpha g^{\beta\gamma}\pi_\gamma \\ &+\frac{1}{2}\left(\pi_\beta g^{\beta\gamma}F^\mu{}_{\alpha\gamma}p_\mu + p_\mu g^{\beta\gamma}F^\mu{}_{\alpha\gamma}\pi_\beta\right) \\ &-\frac{1}{2}p_\mu\left(g^{\mu\sigma}\partial_\sigma A^\nu_\alpha + g^{\nu\sigma}\partial_\sigma A^\mu_\alpha\right)p_\nu + p_M \circ \varphi^M_\alpha - \hat{D}_\alpha U \\ &+\frac{\hbar^2}{4}\hat{D}_\gamma\left(\partial_\alpha g^{\beta\gamma}\partial_\mu A^\mu_\beta\right) - \frac{\hbar^2}{8}\partial_\alpha g^{\beta\gamma}\left(\partial_\mu A^\mu_\beta \partial_\nu A^\nu_\gamma\right) \\ &+\frac{\hbar^2}{4}\partial_\mu\left(F^\mu{}_{\alpha\beta}g^{\beta\gamma}\partial_\nu A^\nu_\gamma\right) + \frac{\hbar^2}{4}\partial_M\left(\eta^{MN}\partial_M\partial_\mu A^\mu_\alpha\right).\end{aligned} \tag{6.15}$$

Equations (6.3) and (6.15) completely describe the motion of a particle on the manifold with an intransitive group of motions. It appears that the dynamics is decomposed in two parts. The first one corresponds to the motion on the orbits of the action of G on M, and the equations of motion have the form of conservation laws for the group momenta $\{p_i : i = \overline{m+1, p}\}$. The second one is related to the motion on M/G, where the dynamics of a particle is governed by Lorentz-type forces. The first one is caused by a non-abelian gauge field A^μ_α with the strength $F^\mu{}_{\alpha\beta}$, that has a purely geometrical origin, while the second one is related to the abelian gauge field φ_α with the strength $\varphi_{\alpha\beta}$ and appears due to the quantum-mechanical nature of the theory (in the classical limit $\hbar \to 0$ this object vanishes). As to terms in (6.15) with the factor $\hbar^2$: these describe rather the operator orderings in the first terms in the r.h.s. of (6.15) than the features of dynamics.

7 Hilbert space of states

Following the scheme introduced in [2] we chose the set of eigenvectors of coordinate operators $\{x^M\}$ as the basis of the Hilbert space for a point particle on M, namely

$$\hat{x}^M|x'\rangle = x'^M|x'\rangle. \tag{7.1}$$

with the following normalization condition:

$$\langle x'\,|x''\rangle = \Delta(x'-x'') := \frac{1}{\sqrt{\eta(x)}}\delta(x'-x''), \tag{7.2}$$

where $\eta(x) = \det(\eta_{MN}(x))$. The Δ-function has the following properties:

$$f(x')\Delta(x'-x'') = f(x'')\Delta(x'-x''), \tag{7.3}$$

$$f(x')\partial'_M\Delta(x'-x'') = -\partial'_M f(x')\Delta(x'-x''), \tag{7.4}$$

$$\begin{aligned}\partial'_M\Delta(x'-x'') &= \partial'_M\left(\frac{1}{\sqrt{\eta(x')}}\Delta(x'-x'')\right) \\ &= -\Gamma_M(x')\Delta(x'-x'') \\ &\quad -\partial''_M\Delta(x'-x'').\end{aligned} \tag{7.5}$$

In terms of the coordinate decomposition $x^M = \{x^\alpha, x^\mu\}$ (7.5) takes the form

$$\begin{aligned}&\left(\hat{D}'_\alpha + \hat{D}''_\alpha + \hat{\Gamma}_\alpha(x')\right)\Delta(x'-x'') \\ &\quad = \partial'_\mu A^\mu_\alpha(x')\Delta(x'-x''),\end{aligned} \tag{7.6}$$

where

$$\hat{D}_\alpha = \partial_\alpha - A^\mu_\alpha\partial_\mu, \quad \hat{\Gamma}_\alpha = \Gamma_\alpha - A^\mu_\alpha\Gamma_\mu.$$

To construct the coordinate representation of the operators describing the quantum mechanics on M, one has to calculate the matrix elements of the coordinate and momentum operators x^M and p_M.

In order to simplify this task we use the previously defined operators

$$\overline{p}_M = \{\overline{p}_\alpha, \overline{p}_\mu\}, \quad \overline{p}_\alpha := p_\alpha - \varphi_\alpha, \quad \overline{p}_\mu := p_\mu \tag{7.7}$$

that obey the following commutation relations:

$$[x^M, x^N] = 0, \quad [x^M, \overline{p}_N] = \mathrm{i}\hbar\delta^M_N, \quad [\overline{p}_M, \overline{p}_N] = 0. \tag{7.8}$$

Using the results of [2] we can write

$$\begin{aligned}&\langle x'|x^M|x''\rangle = \mathrm{i}\hbar x'^M\langle x'|x''\rangle, \\ &\langle x'|\overline{p}_M|x''\rangle = -\mathrm{i}\hbar\left(\partial'_M + \frac{1}{2}\Gamma_M(x')\right)\Delta(x'-x'').\end{aligned} \tag{7.9}$$

Returning to the operators p_M, we find that

$$\begin{aligned}\langle x'|p_\alpha|x''\rangle &= -\mathrm{i}\hbar\left(\partial'_\alpha + \frac{1}{2}\Gamma_\alpha(x')\right)\Delta(x'-x'') \\ &\quad + \varphi_\alpha(x')\Delta(x'-x''), \\ \langle x'|p_\mu|x''\rangle &= -\mathrm{i}\hbar\left(\partial'_\mu + \frac{1}{2}\Gamma_\mu(x')\right)\Delta(x'-x''), \\ \langle x'|x^M|x''\rangle &= x''^M\Delta(x'-x'').\end{aligned} \tag{7.10}$$

Using these expressions we can write the matrix element of the operator π_α in the form

$$\begin{aligned}\langle x'|\pi_\alpha|x''\rangle = &-\mathrm{i}\hbar\left(\hat{D}_\alpha + \frac{1}{2}(\hat{\Gamma}_\alpha - \partial_\mu A^\mu_\alpha)\right)\Delta(x'-x'') \\ &+ \varphi_\alpha(x')\Delta(x'-x'').\end{aligned} \tag{7.11}$$

The coordinate representation of the operator A can be constructed in terms of the wave functions $\psi(x') = \langle x'\,|\psi\rangle$ representing the state $|\psi\rangle$ in the usual way:

$$(\hat{A}\psi)(x') := \int\langle x'|A|x''\rangle\langle x''|\psi\rangle\mathrm{d}x''. \tag{7.12}$$

Therefore, following [2] we find the representations for the coordinate and momentum operators in the following form:

$$\hat{x}^M \psi(x') = x'^M \psi(x'),$$
$$\hat{p}_\alpha \psi(x') = -\mathrm{i}\hbar \left(\partial'_\alpha + \frac{1}{2} \Gamma_\alpha(x') \right) \psi(x') + \varphi_\alpha(x') \psi(x'),$$
$$\hat{p}_\mu \psi(x') = -\mathrm{i}\hbar \left(\partial'_\mu + \frac{1}{2} \Gamma_\mu(x') \right) \psi(x'). \quad (7.13)$$

The coordinate representation of the Hamiltonian can be obtained in terms of the operators

$$\pi_M = p_M - \frac{\mathrm{i}\hbar}{2} \Gamma_M, \quad \pi_M^\dagger = p_M + \frac{\mathrm{i}\hbar}{2} \Gamma_M, \quad (7.14)$$

namely,

$$\hat{H} = \frac{1}{2} \hat{\pi}_M \hat{\eta}^{MN} \hat{\pi}_N^\dagger. \quad (7.15)$$

Using the decomposition of the coordinates of M we can write

$$\pi_\alpha = -\mathrm{i}\hbar \left(\partial_\alpha + \Gamma_\alpha \right) + \varphi_\alpha, \quad \pi_\mu = -\mathrm{i}\hbar \left(\partial_\mu + \Gamma_\mu \right), \quad (7.16)$$
$$\pi_\alpha^\dagger = -\mathrm{i}\hbar \partial_\alpha + \varphi_\alpha, \quad \pi_\mu^\dagger = -\mathrm{i}\hbar \partial_\mu. \quad (7.17)$$

Hence, the formal part of the formulation of the quantum mechanics on the Riemannian manifold with an intransitive group of isometries is solved. Nevertheless, the other one, which contains the meaning and interpretation of these results, requires a special discussion.

8 Discussion

We have examined in the present paper and in [2,3] the generalization of Schwinger's action principle for the case of Riemannian manifolds with a group structure determined by isometric transformations.

The obtained results show a fundamental unity between geometrical properties of a manifold M and the algebra of quantum-mechanical operators describing a point particle on M. The key role in the formulation of quantum mechanics is played by geometrical aspects of a symmetry, expressed in terms of Killing vectors. These vectors also determine the integrals of motion on M and their algebraic properties.

It appears that the general quantization problem is conditionally reduced to three cases. The first one, considered in [2], is related to the class of manifolds with a simply transitive group of isometries. Here the dimension of this group (which coincides with the number of Killing vectors) is exactly equal to the dimension of the manifold. The quantum mechanics gets its traditional form as investigated in [4,5].

The second case contains the investigation of homogeneous manifolds with a non-simple transitive group of isometries, where the dimension of the group exceeds the dimension of the manifold. There occurs a non-abelian gauge structure that is similar to the one in Kaluza–Klein theories. This structure is related to the isotropy subgroup of G and plays the role of a gauge group. This case has been analyzed in [3]. The obtained results are in accordance with [6], where the method of study somewhat differs from ours.

The remaining third case, analyzed in the present paper, is related to Riemannian manifolds with the intransitive group of isometries G. In this case the dimension of the manifold M is higher than the dimension of the group G. From the geometrical point of view M can be considered as the total space of the principal fiber bundles with the base space M/G and the structure group G. In terms of this construction, M can be covered by the set of submanifolds (the orbits of the G action) and G acts simply transitively in each of them. Quantum mechanics on such a submanifold is developed in [2]. The other degrees of freedom, related to the quotient space M/G do not have a uniquely determined dynamics. In order to determine this unequivocally we have to assume something new and external with respect to Schwinger's scheme. There are two types of arbitrary factors in the formulation of quantum mechanics on M. Some of them are of a geometrical nature (the functions b_N^M), while the others are quantum-mechanical objects (the abelian gauge field φ_α that vanishes when $\hbar \to 0$). It is important to point out here that the latter structure cannot be obtained in the framework of a canonical quantization procedure; it is caused by geometrical properties of the manifold and the algebraic relation between the geometrical objects.

The formal results of the present paper lead to the more complicated question of its interpretation. We suppose that this can be done in the following manner. There are some degrees of freedom, describing a quantum system, which have partially undetermined properties (as can be treated in the traditional sense). The geometry of space is determined by energy-momentum of the matter distribution. It is essential to search the connection between geometrical structures corresponding to the undetermined dynamics and forbidden spatial zones for the systems with a discrete energy spectrum (in the manner of "layers" between the Bohr orbits in the hydrogen atom). Nevertheless, the question of the interpretation of quantum theory on a general Riemannian manifold remains an open one and requires a more detailed discussion and further development.

As a logical continuation of our series of papers we thought of devoting attention to the generalization of Schwinger's quantization approach to the case of a supermanifold. We are going to present this in a forthcoming paper.

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